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METHOD OF DISCRETE SINGULARITIES FOR SOLUTION OF SINGULAR INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The common approach, named the method of discrete singularities by Prof. I. K. Lifanov and directed to solution of the integral equations on the finite interval containing Cauchy type singular integral, is proposed. This method can be applied to first and second kind singular integral equations, equations with generalized kernel, as well as to integro-differential equations.

რეზიუმე. წარმოდგენილია ზოგადი მეთოდი, სახელწოდებით ლიფანოვის დისკრეტული სინგულარობის მეთოდი. იგი გამოყენებულია ინტეგრალური განტოლებების ამოხსნისათვის სასრულ ინტერვალზე, რომელიც შეიცავს კოშის ტიპის სინგულარულ ინტაგრალს. ეს მეთოდი შეიძლება გამოყენებულ იქნეს პირველი და მეორე გვარის სინგულარული ინტეგრალური განტოლებებისათვის, ასევე განზოგადოებულ გულიანი და ინტეგრო-დიფერენციალური განტოლებებისათვის.

Introduction

The direct methods on solution of the integral equations are very effective when the obtaining of the closed solution is impossible. The application of Gauss type quadrature formulas on solution of the singular integral equation of first kind is well known, particularly, [1, 2, 3, 4, 5, 6]. Generally, the solutions unbounded at the both ends are considered. In this connection we should note the paper [1]. This work has an important significance, taking into account the wideness of represented material and the successful selection of problems in order to represent the methods of investigation the behavior of solutions at the singularity points and solution of different types of singular integral equations. However, unfortunately, in the marked paper there is no uniform approach in a question of reducing the singular integral equations to system of the algebraic equations. If in the case of first kind equations is successfully applied method of mechanical quadratures, then in the case of second kind equations is presented other method – method of

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orthogonal polynomials. Method of mechanical quadratures presented for equations with generalized Cauchy kernel has formal character.

The method of discrete singularities is the universal method for solution of singular integral equations on finite interval when unknown function can be represented as product of bounded continuous function and weight function $(1-t)^{\alpha}(1+t)^{\beta}$ $(\operatorname{Re}(\alpha,\beta)>-1)$. The method is based on the quadrature formula of the highest algebraic precision for singular integral. The application of the method of discrete singularities on solution of singular integral equations of the second kind with real coefficient of the free member is shown and tested in the papers [7, 8], with complex coefficient – in the papers [9, 10], and equations with generalized Cauchy kernel in the papers [11, 12]. It is necessary to note that the system of algebraic equations obtained by the method of discrete singularities for the first kind singular integral equations completely coincides with the system in the paper [1].

1. Quadrature Formulas of the Highest Algebraic Precision for Singular Integrals

In the papers [13, 14, 15] the following quadrature formulas were obtained:

a)

$$\int_{-1}^{1}\frac{\varphi^{*}(x)}{x-z}(1-x)^{\alpha}(1+x)^{\beta}\,dx\approx\sum_{i=1}^{n}w_i\frac{\varphi^{*}(\xi_i)}{\xi_i-z}\left[1-\frac{R_n^{(\alpha,\beta)}(z)}{R_n^{(\alpha,\beta)}(\xi_i)}\right] \tag{1}$$

$$(z\neq\pm1,\ \operatorname{Re}\alpha,\operatorname{Re}\beta>-1),$$

where the nodes ξ_i $\left(i=\overline{1,n}\right)$ are roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, $\varphi^{*}(x)$ is continuous and bounded on closed interval $[-1,1]$, weights w_i are defined by formula

$$w_i=\frac{2}{n+\alpha+\beta+1}\,\frac{R_n^{(\alpha,\beta)}(\xi_i)}{P_{n-1}^{(\alpha+1,\beta+1)}(\xi_i)} \tag{2}$$

and

$$R_n^{(\alpha,\beta)}(z)=\begin{cases}-\left(\dfrac{2}{z-1}\right)^{n+1}2^{\alpha+\beta}B(n+\alpha+1,n+\beta+1)\times\\ \qquad\times F\left(n+1,n+\alpha+1;2n+\alpha+\beta+2;\dfrac{2}{1-z}\right),\\ \dfrac{R_n^{(\alpha,\beta)}(z+i0)+R_n^{(\alpha,\beta)}(z-i0)}{2}\qquad(-1<z<1).\end{cases} \tag{3}$$

The $R_n^{(\alpha,\beta)}(z)$ has many different representations and one of them is (3).

The following formula (4) for characteristic part of a singular integral equation of the second kind

b)

$$\int_{-1}^{1} \frac{\varphi(x)}{x-y}dx + \pi\lambda\varphi(y) \approx \sum_{i=1}^{n} \frac{w_i\,\varphi^*(\xi_i)}{\xi_i - y}\left[1 - \frac{P_{n+\kappa}^{(-\alpha,-\beta)}(y)}{P_{n+\kappa}^{(-\alpha,-\beta)}(\xi_i)}\right] \qquad (4)$$

is obtained by using representation

$$R_n^{(\alpha,\beta)}(z) = \frac{\pi(z-1)^{\alpha}(z+1)^{\beta}}{\sin\pi\alpha}P_n^{(\alpha,\beta)}(z) - \\ - 2^{\alpha+\beta}B(\alpha, n+\beta+1)\,F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-z}{2}\right).$$

Here α and β are found from equations

$$\begin{matrix} \operatorname{ctg}\pi\alpha + \lambda = 0 \\ \operatorname{ctg}\pi\beta - \lambda = 0 \end{matrix}, \qquad \kappa = \alpha + \beta = \begin{cases} -1 \\ \ \ 0 \\ \ \ 1 \end{cases}, \qquad (5)$$

$$\varphi(x) = \varphi^*(x)\,\omega(x), \qquad \omega(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

Weights w_i are defined by (2). But in case when the conditions (5) are satisfied, (2) can be represented in simpler form

$$w_i = -\frac{2^{\kappa+1}\pi}{(n+\kappa+1)\sin\pi\alpha}\,\frac{P_{n+\kappa}^{(-\alpha,-\beta)}(z_i)}{P_{n-1}^{(\alpha+1,\beta+1)}(z_i)}. \qquad (6)$$

It is necessary to note that quadrature formula (4) is valid for both real and complex coefficient λ.

c) Let us consider the following integral

$$J(z) = \int_{-1}^{1}\frac{\varphi'(x)}{x-z}dx = \int_{-1}^{1}\frac{d}{dx}\left[\varphi^*(x)\,\omega(x)\right]\frac{dx}{x-z} \qquad (7)$$

$$(z \in C, z \neq \pm 1, \quad \mathrm{Re}\,[\alpha,\beta] \geq 0).$$

In case $\alpha \neq 0,\ \beta \neq 0$, we have

$$J(z) \approx -\frac{2}{n+\alpha+\beta+1}\sum_{j=1}^{n}\frac{\varphi^*(\xi_j)}{P_{n-1}^{(\alpha+1,\beta+1)}(\xi_j)}\times \\ \times\left[\frac{2(n+1)}{z-\xi_j}R_{n+1}^{(\alpha-1,\beta-1)}(z) + \frac{R_n^{(\alpha,\beta)}(z) - R_n^{(\alpha,\beta)}(\xi_j)}{(z-\xi_j)^2}\right], \qquad (8)$$

when $\alpha = 0,\ \beta \neq 0$

$$J(z) \approx \frac{2}{n+\beta+1} \sum_{j=1}^{n} \frac{\varphi^*(\xi_j)}{P_{n-1}^{(1,\beta+1)}(\xi_j)} \left\{ \frac{-2^{\beta}}{(1-\xi_j)(z-1)} - \right.$$
$$\left. -(n+\beta) \frac{R_n^{(1,\beta-1)}(z)}{(z-\xi_j)(z-1)} - \frac{R_n^{(0,\beta)}(z) - R_n^{(0,\beta)}(\xi_j)}{(z-\xi_j)^2} \right\}, \qquad (9)$$

when $\alpha \neq 0,\ \beta = 0$

$$J(z) \approx \frac{2}{n+\alpha+1} \sum_{j=1}^{n} \frac{\varphi^*(\xi_j)}{P_{n-1}^{(\alpha+1,1)}(\xi_j)} \left\{ \frac{-2^{\alpha}}{(1+\xi_j)(z+1)} - \right.$$
$$\left. -(n+\alpha) \frac{R_n^{(\alpha-1,1)}(z)}{(z-\xi_j)(z+1)} - \frac{R_n^{(\alpha,0)}(z) - R_n^{(\alpha,0)}(\xi_j)}{(z-\xi_j)^2} \right\}, \qquad (10)$$

when $\alpha = 0,\ \beta = 0$

$$J(z) \approx \frac{2}{n+1} \sum_{j=1}^{n} \frac{\varphi^*(\xi_j)}{(z-\xi_j) P_{n-1}^{(1,1)}(\xi_j)} \left[\frac{n+1}{2} \frac{R_{n-1}^{(1,1)}(z)}{1-z^2} - \right.$$
$$- \frac{R_n^{(0,0)}(z) - R_n^{(0,0)}(\xi_j)}{z-\xi_j} +$$
$$\left. + \frac{z-\xi_j}{(1-z)(1-\xi_j)} \left[1 - (-1)^n + \frac{2(-1)^n (z+\xi_j)}{(1+z)(1+\xi_j)} \right] \right]. \qquad (11)$$

d) The quadrature formula for the following integral is also of interest

$$I(y) = \int_{-1}^{y} \varphi(x)\, dx = \int_{-1}^{y} \varphi^*(x) (1-x)^{\alpha} (1+x)^{\beta}\, dx \qquad (12)$$
$$(|y| \leq 1, \quad \mathrm{Re}\,[\alpha], \mathrm{Re}\,[\beta] > -1).$$

This one is

$$I(y) \approx -\frac{2}{n+\alpha+\beta+1} \frac{k_{n+1} h_n}{k_n} \sum_{i=1}^{n} \frac{a_i(y)\, \varphi^*(\xi_i)}{P_{n-1}^{(\alpha+1,\beta+1)}(\xi_i)\, P_{n+1}^{(\alpha,\beta)}(\xi_i)}, \qquad (13)$$

where

$$a_i\left(y\right)=\left[\frac{\left(1+y\right)^{\beta+1}\Gamma\left(\alpha+\beta+2\right)}{2^{\beta+1}\Gamma\left(1+\alpha\right)\Gamma\left(2+\beta\right)}F\left(\beta+1,\ -\alpha;\ 2+\beta;\frac{1+y}{2}\right)-\right.$$

$$\left.-\left(1-y\right)^{\alpha+1}\left(1+y\right)^{\beta+1}\sum_{m=1}^{n-1}\frac{P_m^{(\alpha,\beta)}\left(\xi_i\right)P_{m-1}^{(\alpha+1,\beta+1)}\left(y\right)}{2m\,h_m}\right],$$

$$k_m=\frac{\Gamma\left(2m+\alpha+\beta+1\right)}{2^m\Gamma\left(m+\alpha+\beta+1\right)\Gamma\left(m+1\right)},$$

$$h_m=\frac{2^{\alpha+\beta+1}\Gamma\left(m+\alpha+1\right)\Gamma\left(m+\beta+1\right)}{\left(2m+\alpha+\beta+1\right)\Gamma\left(m+1\right)\Gamma\left(m+\alpha+\beta+1\right)}.$$

2. Singular Integral Equations of the Second Kind

We recall that singular integral equations of the second kind have the following form

$$\int_{-1}^{1}\frac{\varphi\left(x\right)}{x-y}dx+\pi\lambda\varphi\left(y\right)+\int_{-1}^{1}K\left(x,y\right)\varphi\left(x\right)dx=f\left(y\right)\quad\left(-1<y<1\right).\tag{14}$$

Here $\varphi\left(x\right)$ is unknown, λ is constant, which can be real or complex, the known functions $K\left(x,y\right)$ and $f\left(y\right)$ are H – continuous.

The solution of (14) can be represented in the form

$$\varphi\left(x\right)=\varphi^*\left(x\right)\left(1-x\right)^{\alpha}\left(1+x\right)^{\beta}\quad\left(\mathrm{Re}\left[\alpha\right],\mathrm{Re}\left[\beta\right]>-1\right),\tag{15}$$

where $\varphi^*\left(x\right)$ is a bounded continuous function in the closed interval [-1,1].

Using the well-known results of N.I.Muskhelishvili [16] about behavior of Cauchy integral near the end points of integration interval, we will find the exponents α and β. They satisfy the equations (5).

On the base of (4) and known quadrature formula for regular integral, i.e.

$$\int_{-1}^{1}K\left(x,y\right)\varphi^*\left(x\right)\omega\left(x\right)dx\approx\sum_{i=1}^{n}w_i\,K\left(\xi_i,y\right)\varphi^*\left(\xi_i\right)\tag{16}$$

equation (14) can be rewriting in the form:

$$\sum_{i=1}^{n}w_i\,\varphi^*\left(\xi_i\right)\left[\frac{1}{\xi_i-y}\left(1-\frac{P_{n+\kappa}^{(-\alpha,-\beta)}\left(y\right)}{P_{n+\kappa}^{(-\alpha,-\beta)}\left(\xi_i\right)}\right)+K\left(\xi_i,y\right)\right]=f\left(y\right).\tag{17}$$

For finding the unknown functions $\varphi^*\left(\xi_i\right)\ \left(i=\overline{1,n}\right)$ we must equate both parts of (17) in definite number of collocation points. It is convenient to choose the roots of $P_{n+\kappa}^{(\alpha,\beta)}\left(y\right)$ as a set of collocation points. Since index κ can get three different values we will consider every case separately.

Let $\kappa = -1$. Then the roots of $P_{n-1}^{(\alpha,\beta)}(y)$ are set of collocation points, i.e. we have only $n-1$ equations for definition n unknowns. But as it is known, in this case equation (14) hasn't a unique solution and we need in additional condition, which usually is given in the form:

$$\int_{-1}^{1} \varphi^*(x)\,\omega(x)\,dx = C. \tag{18}$$

After discretization of (18), we will obtain the following closed system of linear algebraic equations with respect to $\varphi^*(\xi_i)$ $\left(i = \overline{1,n}\right)$

$$\begin{aligned} &\sum_{i=1}^{n} w_i\,\varphi^*(\xi_i)\left[\frac{1}{\xi_i - z_k} + K(\xi_i, z_k)\right] = f(z_k) \quad \left(k = \overline{1,n-1}\right), \\ &\sum_{i=1}^{n} w_i\,\varphi^*(\xi_i) = C, \end{aligned} \tag{19}$$

where z_k are the roots of $P_{n-1}^{(\alpha,\beta)}(y)$.

Let $\kappa = 0$. In this case the set of collocation points will be the roots t_k of polynomial $P_n^{(\alpha,\beta)}(y)$ and we will have the closed system of linear algebraic equations

$$\sum_{i=1}^{n} w_i\,\varphi^*(\xi_i)\left[\frac{1}{\xi_i - t_k} + K(\xi_i, t_k)\right] = f(t_k) \quad \left(k = \overline{1,n}\right). \tag{20}$$

Let $\kappa = 1$. Now we will have $n+1$ equations for finding n unknowns. Here also it is known that equation (14) has solution only if the following consistency condition will be satisfied:

$$\int_{-1}^{1}\left[\int_{-1}^{1} K(x,y)\,\varphi^*(x)\,\omega(x)\,dx - f(y)\right]\frac{dy}{(1-y)^{\alpha}(1+y)^{\beta}} = 0. \tag{21}$$

According [17] the additional unknown γ_{0n} is introduced in the system of linear algebraic equations

$$\gamma_{0n} + \sum_{i=1}^{n} w_i\,\varphi^*(\xi_i)\left[\frac{1}{\xi_i - \zeta_k} + K(\xi_i, \zeta_k)\right] = f(\zeta_k) \quad \left(k = \overline{1,n+1}\right). \tag{22}$$

The closed system is obtained. The unknown γ_{0n} is a regularizing factor, because $\gamma_{0n} \to 0$ as $n \to \infty$ only in case when the condition (21) is satisfied. The presence of such regularizing factor is very convenient when the satisfaction the condition (21) depends from any parameter that must be appropriately chosen.

3. Singular Integral Equations of the First Kind

Singular integral equations of the first kind is

$$\int_{-1}^{1} \frac{\varphi(x)}{x-y} dx + \int_{-1}^{1} K(x,y)\,\varphi(x)\,dx = f(y) \quad (-1 < y < 1). \tag{23}$$

Here $K(x,y)$ and $f(y)$ are known continuous functions of its arguments.

From the well-known results of N.I.Muskhelishvili [16] about behavior of Cauchy integral near the end points of integration interval, it is found that $\alpha = \pm\beta = \pm 0.5$. These values depend on problem statement.

It is obvious that putting $\lambda = 0$ in second kind equations the same results obtained above can be used for first kind equations.

To make the picture complete the corresponding systems will be represented:
when $\alpha = \beta = -0.5$

$$\begin{aligned} &\frac{\pi}{n}\sum_{i=1}^{n} \varphi^*(\xi_i)\left[\frac{1}{\xi_i - z_k} + K(\xi_i, z_k)\right] = f(z_k) \quad \left(k = \overline{1, n-1}\right), \\ &\frac{\pi}{n}\sum_{i=1}^{n} \varphi^*(\xi_i) = C, \end{aligned} \tag{24}$$

$$\xi_i = \cos\frac{(2i-1)\pi}{2n} \quad (i = 1, 2, \ldots, n), \quad z_k = \cos\frac{k\pi}{n} \quad (k = 1, 2, \ldots, n-1),$$

when $\alpha = -\beta = \pm 0.5$

$$\sum_{i=1}^{n} w_i\,\varphi^*(\xi_i)\left[\frac{1}{\xi_i + \xi_k} + K(\xi_i, -\xi_k)\right] = f(-\xi_k) \quad \left(k = \overline{1, n}\right), \tag{25}$$

$$w_j = \mp\frac{2\pi}{n+1}\frac{P_n^{(-\alpha,\alpha)}(\xi_j)}{P_{n-1}^{(\alpha+1,1-\alpha)}(\xi_j)}, \quad \xi_i = \pm\cos\frac{2i\pi}{2n+1} \quad (i = 1, 2, \ldots, n),$$

when $\alpha = \beta = 0.5$

$$\gamma_{0n} + \sum_{i=1}^{n} w_i\,\varphi^*(\xi_i)\left[\frac{1}{\xi_i - z_k} + K(\xi_i, z_k)\right] = f(z_k) \quad \left(k = \overline{1, n+1}\right), \tag{26}$$

$$w_j = -\frac{\pi}{n+1}\frac{\left(1 - \xi_j^2\right) T_{n+1}(\xi_j)}{U_{n-1}(\xi_j)}, \quad \xi_j = \cos\frac{j\pi}{n+1} \quad (j = 1, 2, \ldots, n).$$

The system (24) one can meet in [1, 2, 3, 4, 5, 6] as well in many other works.

4. Singular Integral Equations with Generalized Cauchy Kernel

The singular integral equation with Cauchy generalized kernel has the following form

$$\int_{-1}^{1} \frac{\varphi(x)}{x-y} dx + \int_{-1}^{1} k(x,y)\,\varphi(x)\,dx + \int_{-1}^{1} K(x,y)\,\varphi(x)\,dx = f(y)\,, \qquad (27)$$

where $k(x,y)$ is determined, in general, by the formula [1]

$$k(x,y) = \sum_{0}^{N} c_k (y+1)^k \frac{d^k}{dy^k} (x-z_1)^{-1} + \\ + \sum_{0}^{M} b_j (1-y)^j \frac{d^j}{dy^j} (x-z_2)^{-1}\,, \qquad (28)$$

$$z_1 = -1 + (y+1)\,e^{i\theta_1}, \quad z_2 = 1 + (1-y)\,e^{i\theta_2}$$
$$-1 < (y,x) < 1, \quad 0 < \theta_1 < 2\pi, \quad -\pi < \theta_2 < \pi$$

Using the well-known results of N. I. Muskhelishvili [16] about behavior of Cauchy type integral near the end points of integration interval for two first terms in equation (27), we get two independent transcendental equations with respect to exponents α and β. One of them contains only coefficients c_k and other one contains only coefficients b_j. After finding the exponents α and β the solution can be represented in form (15). It is obvious that in this case the sum of α and β is not integer and $R_n^{(\alpha,\beta)}(z)$ is not polynomial. In this case the choice of the set of collocation points is not clear than in above considered cases.

The described method was applied for solution of numerous mixed boundary value problems. The solutions of considered problems were reduced to either one equation of type (27) or system containing two such kind equations. The numerical analysis of these equations displayed that the number of collocation points is determined by number of $R_n^{(\alpha,\beta)}(z)$ roots. It is reasonable that the set of collocation points would be roots of $R_n^{(\alpha,\beta)}(z)$, but the roots of a polynomial of corresponding order can be chosen as well. When number of $R_n^{(\alpha,\beta)}(z)$ roots is less than n, we need additional conditions.

Because the formula (1) is true for whole complex plane excluding only points $z = \pm 1$, it can be also used for first terms of series in (28), but derivatives of (1) with respect to z can be used for other terms.

Thus we will obtain the closed system of linear algebraic equations with respect to $\varphi^*(\xi_i)$ $\left(i = \overline{1,n}\right)$.

5. Singular Integro-Differential Equations

We will have a singular integro-differential equation when equations (14), (23) or (27) will contain an additional term in the form of integral $J(z)$ from (7) or $I(z)$ from (12). Since structure of quadrature formula for mentioned integrals is similar to the formulas for main terms in equations, it is not difficult to obtain corresponding systems of linear algebraic equations.

For example, let us consider the following equation

$$\int_{-1}^{1} \frac{\varphi(s)}{s-x} ds + \lambda \int_{-1}^{x} \varphi(s)\, ds = F(x) \qquad (-1 < x < 1), \tag{29}$$

which is a governing equation of a plane problem for semi-plane with elastic stringer [18].

Let unknown function is unbounded at the both ends. In this case we will need in additional condition

$$\int_{-1}^{1} \varphi(s)\, ds = C. \tag{30}$$

It is not difficult to show that solution of (29) can be represented in the form

$$\varphi(x) = \frac{\varphi^*(x)}{\sqrt{1-x^2}}, \tag{31}$$

where $\varphi^*(x)$ is bounded continuous function.

Taking into account (31), from (1) and (13) we get

$$\int_{-1}^{1} \frac{\varphi(s)}{s-x} ds = \int_{-1}^{1} \frac{1}{s-x} \frac{\varphi^*(s)}{\sqrt{1-s^2}} ds \approx \frac{\pi}{n} \sum_{j=1}^{n} \frac{\varphi^*(\xi_j)}{\xi_j - x} \left[1 - \frac{U_{n-1}(x)}{U_{n-1}(\xi_j)}\right],$$

$$\int_{-1}^{x} \frac{\varphi^*(s)}{\sqrt{1-s^2}} ds = \frac{1}{n} \sum_{j=1}^{n} \varphi^*(\xi_j) \Bigg[\arcsin x + \frac{\pi}{2} - $$

$$ -2\sqrt{1-x^2} \sum_{m=1}^{n-1} \frac{T_m(\xi_j)\, U_{m-1}(x)}{m} \Bigg], \quad T_n(\xi_i) = 0.$$

Choosing the roots of polynomial $U_{n-1}(x)$ as set of collocation points, as well as adding the discretization of condition (30), the following system is obtained:

$$\frac{1}{n}\sum_{i=1}^{n}\varphi^*\left(\xi_i\right)\left[\frac{\pi}{\xi_i - x_k} + \lambda\left(\arcsin x_k + \frac{\pi}{2} - \right.\right.$$
$$\left.\left. -2\sqrt{1-x_k^2}\sum_{m=1}^{n-1}\frac{T_m\left(\xi_i\right)U_{m-1}\left(x_k\right)}{m}\right)\right] = F\left(x_k\right), \qquad (32)$$
$$\frac{\pi}{n}\sum_{i=1}^{n}\varphi^*\left(\xi_i\right) = C.$$

It is clear that in the same way it is possible to get a corresponding system for other equations too.

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